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My research is in **algebraic combinatorics**, **representation theory**, and the interactions between them. There is a symbiotic relationship between these two fields, with advances in one area often leading to discoveries in the other. This is illustrated by the interplay between the combinatorial family of Macdonald polynomials and the representation theory of reductive groups and double affine Hecke algebras, which we will discuss shortly. My thesis research builds upon these connections to give a combinatorial formula for an exciting and recently discovered family of polynomials generalizing Macdonald polynomials. In ongoing joint work, I am also studying a different instance of the interaction between algebraic combinatorics and representation theory: namely, the symbiosis between partition identities and the theory of vertex operator algebras. In what follows, I will give a brief overview of these two projects, then give further details regarding my thesis research and its future directions.

Macdonald polynomials and double affine Hecke algebras. The representation theory of reductive groups gives rise to a wide variety of special functions, such as the appearance of Schur polynomials as characters of the general linear group. Around 1988, Macdonald [M1]-[M3] made the astonishing observation that many of these special functions can be generalized by a single family of polynomials, now called (symmetric) *Macdonald polynomials*. These led to the construction of, and are profitably studied using, the *double affine Hecke algebra* (DAHA) and its *polynomial representation*, which were first introduced in the work of Cherednik, [C1]-[C3]. This representation of the DAHA was then used to define the *nonsymmetric Macdonald polynomials* [O,C4,M4], which have many of their own interesting combinatorial and representation-theoretic properties. Sahi, Stokman, and Venkateswaran [SSV,SSV2] have recently constructed the *metaplectic representation* of the DAHA, leading to the new, more general family of *SSV polynomials*. These discoveries are naturally connected to objects in number theory: the metaplectic representation was motivated by the work of [CG1,CG2] on Weyl group multiple Dirichlet series, and SSV polynomials specialize to certain Whittaker functions related to metaplectic covers of reductive p-adic groups (this observation is due to [SSV2] and is sketched in [Sa]). My work is motivated by the following questions:

Problem 1. (a) How can we use the metaplectic representation of the DAHA to understand the combinatorics of SSV polynomials?

(b) What representation-theoretic interpretations do SSV polynomials have? What insights about representation theory can we gain from their combinatorial structure?

In [Sa], I addressed Problem 1(a), using the metaplectic representation and a result of [RY] to prove the first combinatorial formula for SSV polynomials. My ongoing work applies this formula to prove an analogue of the Littlewood-Richardson rule for the product of two SSV polynomials, generalizing a result of [Y]. Below, after giving more details about the relevant background and my contributions, I will discuss future plans for more fully solving Problem 1, including potential connections to solvable lattice models.

Vertex operator algebras and partition identities. The *Rogers-Ramanujan* (RR) identities are a famous pair of q-series identities, with the form "infinite sum = infinite product," that can be expressed as partition identities. Two among the many notable proofs of the RR identities are the "motivated proof" of Andrews and Baxter [AB] (which they described as "essentially equivalent" to the proof of [RR]) and the vertex operator theoretic proof of Lepowsky and Wilson [LW1,LW2]. These inspired Lepowsky and A. Milas to propose that the motivated proof, which "intertwines" the two RR identities, might have an interpretation by means of suitable vertex operator algebraic intertwining operators:

Problem 2. Construct a representation-theoretic "categorification" of the motivated proof of the Rogers-Ramanujan identities by means of suitable twisted intertwining operators in the theory of generalized vertex operator algebras.

In ongoing work, A. Ginory, S. Kanade, J. Lepowsky, and I [GKLS] have solved a version of Problem 2 for a much simpler but analogous pair of q-series identities of Euler, which were given a "motivated proof" in [KLRS] as a special case of an infinite family of such proofs. (The Introduction in [KLRS] includes a partial overview of an extensive program toward the solution of Problem 2.) This has led to interesting discoveries about twisted intertwining operators for generalized vertex operator algebras, which we believe will lead us to a solution of Problem 2 and its analogue for generalized RR identities.

1 Macdonald Polynomials and Double Affine Hecke Algebras

1.1 Background and Motivation

Let r be a positive integer, and let P be the weight lattice corresponding to GL_r, which we identify with $\mathbb{Z}^r$. (We will focus on the GL_r case, as it is the situation studied in [Sa] and the final section of [SSV], but similar results can be proven for arbitrary reductive groups. This will be studied in my thesis.) We consider the *double affine Hecke algebra* (DAHA) $\mathbb{H}$, an associative algebra whose generators include elements $T_1, \ldots, T_{r-1}$, X^ν, and Y^ν for $\nu \in P$. The *polynomial representation* is a representation π of $\mathbb{H}$ on the space of Laurent polynomials, $\mathbb{F}[P] = \mathrm{span}_{\mathbb{F}}\{x^\mu : \mu \in P\}$, where $\mathbb{F} = \mathbb{C}(k, q)$ for independent parameters k and q. (The DAHA and its polynomial representation were first introduced by Cherednik in [C1]-[C3], and a detailed exposition is given in [C6].) The operators $\pi(Y^\nu)$, $\nu \in P$, commute and are simultaneously diagonalizable. The *nonsymmetric Macdonald polynomials* $E_\mu = E_\mu(x; q, k)$, $\mu \in P$, are the common eigenfunctions of the operators $\pi(Y^\nu)$ (normalized to have leading coefficient 1). These polynomials, which can be used to recover the symmetric Macdonald polynomials, were introduced in successively greater levels of generality by Opdam [O], Macdonald [M4], and Cherednik [C4]. Nonsymmetric Macdonald polynomials have a plethora of representation-theoretic interpretations: see, for instance, [BBL, CO, I1, I2] and the detailed inventory in [OS].

In order to study the local parts of Weyl group multiple Dirichlet series, Chinta and Gunnells [CG1, CG2] introduced a new action of the Weyl group of the reductive group on a space of rational functions, generalizing a result of Kazhdan and Patterson in type A [KP]. This action depends on a positive integer n. The proof that this was indeed an action of the Weyl group relied on a computer check. Sahi, Stokman, and Venkateswaran [SSV, SSV2] provided a conceptual proof using the technique of "Baxterization" (see [C6]). In order to do so, they considered a DAHA $\mathbb{H}^{(n)}$ that is isomorphic to $\mathbb{H}$.[1] They constructed the *metaplectic representation* $\pi^{(n)}$ of $\mathbb{H}^{(n)}$ on $\mathbb{F}^{(n)}[P]$ (where $\mathbb{F}^{(n)}$ is an extension of $\mathbb{F}$ with additional parameters G_i, $1 \le i \le \lfloor \frac{n}{2} \rfloor$) in a natural way involving induced representations. They then used the isomorphism between localizations of the affine Weyl group and the affine Hecke algebra $\mathbb{H}_Y^{(n)} \subset \mathbb{H}^{(n)}$ to recover the Weyl group action of [CG1, CG2]. Similarly to the Macdonald case, the operators $\pi^{(n)}(Y^{n\nu})$ for $\nu \in P$ commute and are simultaneously diagonalizable, and their simultaneous eigenfunctions are the *SSV polynomials* $E_\mu^{(n)} = E_\mu^{(n)}(x; q, k)$, $\mu \in P$ [SSV, SSV2]. The SSV polynomials also depend on the G_i parameters, although we omit them from the notation. For $\mu = n\lambda$, $\lambda \in P$, the SSV polynomial $E_\mu^{(n)}(x; q, k)$ is the nonsymmetric Macdonald polynomial $E_\lambda(x^n; q^n, k)$. In particular, for $n = 1$, we have $E_\mu^{(1)} = E_\mu$. Further, when the G_i parameters are specialized to certain Gauss sums and $q \to 0$ or $q \to \infty$, SSV polynomials recover Iwahori Whittaker functions for metaplectic n-fold covers of reductive p-adic groups, generalizing a result of [BBL] in the Macdonald case.[2]

[1] Outside of type A, the analogue of $\mathbb{H}^{(n)}$ could be isomorphic to $\mathbb{H}$ or the DAHA for the dual root system. In the type A setting, $\mathbb{H}^{(n)}$ can be thought of as a rescaling of $\mathbb{H}$, in which $X^\nu \mapsto X^{n\nu}$ and $Y^\nu \mapsto Y^{n\nu}$ for $\nu \in P$. One can work with $\mathbb{H}$ instead, but then the representation and polynomials we discuss below will involve elements of $\frac{1}{n}P$ rather than P. This is the approach taken in [SSV2], where it is generalized even further.

[2] This observation, due to Sahi, Stokman, and Venkateswaran, is sketched and stated in [Sa] and proved more

The crucial property of nonsymmetric Macdonald polynomials is that they satisfy a recursion coming from the *intertwiners*, certain special elements in a localization of a subalgebra of $\mathbb{H}$. The importance of the intertwiners as creation operators for Macdonald polynomials was first pointed out in the papers of Knop and Sahi [Kn, S] for GL_r, building on earlier work [KS] on the Jack limit. These ideas were extended by Cherednik [C5] for arbitrary reduced root systems and Sahi [S2] for type BC. In particular, every Macdonald polynomial may be obtained by applying suitable products of intertwiners to the Macdonald polynomial $E_0 = 1$. In [RY], Ram and Yip gave an expansion formula for such products of intertwiners in terms of combinatorial objects called alcove walks, certain sequences of alcoves that correspond to words in the generators of the affine Weyl group. By applying these intertwiners to 1 and using their intertwiner product formula, Ram and Yip obtained an alcove walk formula for Macdonald polynomials.

1.2 Results

In [Sa], I generalized the alcove walk formula of [RY] to the case of SSV polynomials. Let

$$A^{(n)} = \{(\lambda_1, \dots, \lambda_r) \in P : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r, \lambda_1 - \lambda_r \leq n\},$$

an analogue of the closed fundamental alcove. Then we have

Theorem 1.1. [Sa] *$\mu \in \mathbb{Z}^r$ uniquely determines $\lambda \in A^{(n)}$ and an affine Weyl group element w such that*

$$E_\mu^{(n)} = \sum_{p \in \mathcal{B}(\vec{w})} a^{(n)}(p) x^{n \mathrm{wt}(p) + \phi(p)\lambda}.$$

The coefficients $a^{(n)}(p) \in \mathbb{F}^{(n)}$ are given explicitly; $\mathcal{B}(\vec{w})$ is a set of alcove walks determined by a reduced expression for w; and $wt(p) \in P$ and $\phi(p) \in W_0$ (where W_0 is the finite Weyl group) are determined by the last alcove in the alcove walk p.

This result is proved similarly to the alcove walk formula of Ram and Yip [RY]. These techniques can be adapted because the SSV polynomials can be constructed using the action of the intertwiners via the metaplectic representation. However, a key difference from the Macdonald case is that, rather than applying the intertwiners to $E_0 = 1$, one needs to apply them to all x^λ, $\lambda \in A^{(n)}$, creating some additional subtleties.

As consequences of this result, I obtained alcove walk formulas for symmetrized SSV polynomials (generalizing symmetric Macdonald polynomials) and for certain specializations that relate to metaplectic Iwahori Whittaker functions. Further, a detailed analysis of the terms in Theorem 1.1 gave the following triangularity result, explicitly characterizing the support of the SSV polynomials:

Theorem 1.2. [Sa] *For $\mu \in P$,*

$$E_\mu^{(n)} = x^\mu + \sum_{\nu \overset{n}{<} \mu} c_\nu x^\nu$$

for some nonzero *scalars $c_\nu \in \mathbb{F}^{(n)}$, where $\overset{n}{<}$ is a partial ordering on P related to the Bruhat order.*

I then used this result and some calculations involving the partial order $\overset{n}{<}$ to prove the following corollary, where the *support* of a Laurent polynomial f is the set of all $\mu \in P$ such that x^μ appears in the monomial expansion of f:

Corollary 1.3. [Sa] *Let $\mu \in P$, and suppose m and n are positive integers with $m|n$. Then the support of $E_\mu^{(n)}$ is a subset of the support of $E_\mu^{(m)}$.*

This allows us to understand how the support of $E_\mu^{(n)}$ changes as n varies. In particular, we see that for $n > 1$, the SSV polynomial $E_\mu^{(n)}$ has "fewer terms" than the corresponding Macdonald polynomial $E_\mu^{(1)}$.

generally in [SSV2].

1.3 Future Work

There are several very interesting directions in which to take this project.

1. In my thesis, I plan to generalize the above results to the setting of SSV polynomials associated with a general reductive group (as opposed to the type A case considered above).

2. One of the first applications of the alcove walk formula of [RY] was Yip's result giving a Littlewood-Richardson rule for Macdonald polynomials [Y]. In work in progress, I am extending this to the case of SSV polynomials. This relates to Problem 1(b): for instance, since SSV polynomials specialize to certain Whittaker functions, we obtain a Littlewood-Richardson rule for those Whittaker functions.

3. The alcove walk formula of Theorem 1.1 gives an answer to Problem 1(a), but it is only one of many possible answers. There are two avenues that I find particularly interesting:

   (a) SSV polynomials generalize both nonsymmetric Macdonald polynomials and metaplectic Iwahori Whittaker functions. Both of these families have been constructed via solvable lattice models: these models were given in [BW, ABW] for Macdonald polynomials and [BBBG] for metaplectic Iwahori Whittaker functions. I am interested in finding a common generalization of these models to the setting of SSV polynomials. I believe that this will require a more in-depth understanding of the representation theory of the quantum groups associated to affine Lie superalgebras. These lattice models are constructed using certain solutions to the Yang-Baxter equation, which typically arise from R-matrices of quantum groups. However, the solution to the Yang-Baxter equation studied in [BBBG] currently has no known quantum group interpretation and is in fact proved via a computer check. I would like to understand the results of [BBBG] through the lens of quantum groups, then apply this understanding to generalize the corresponding lattice model to the case of SSV polynomials.

   (b) Another well-known combinatorial formula for Macdonald polynomials is the non-attacking filling formula of [HHL]. This formula was connected to Ram and Yip's alcove walk formula in [Le, GR]. I would like to give an analogue of these arguments in the SSV setting, hopefully arriving at a new formula generalizing that of [HHL]. A potential avenue of attack is suggested by the recent work [LeS], which connects the alcove walk, HHL, and Tokuyama-style formulas for type A Whittaker functions. I would like to study the generalization of these results to the metaplectic setting, which should provide inspiration for the more general SSV case.

4. Problem 1(b) remains open. I know of no representation-theoretic interpretations of proper SSV polynomials (that is, those that are not also Macdonald polynomials) other than those that arise through the connection to metaplectic Iwahori Whittaker functions. I would like to investigate the extent to which certain results, such as the correspondence of certain specializations of Macdonald polynomials with affine Demazure characters [San, I1] and zonal spherical functions (see [M3]), extend to SSV polynomials.

2 Other Projects

During the summer of 2021, I studied a problem related to the Wigner function formalism in quantum mechanics with Dr. A. F. Barghouty and F. P. Eblen at NASA Headquarters. With Dr. E. Rieffel at the NASA Ames Research Center, I am working on a problem in quantum computation, which we expect to lead to a publication.

As an undergraduate, I participated in an REU project with B. A. Itzá-Ortiz, M. B. Malachi, A. Marstaller, and S. Underwood. Afterward, we published the work [IMMSU], in which we classified “eventually periodic” subshifts in the field of symbolic dynamics up to conjugacy and flow equivalence.

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